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A Note on a Paper by Atkin and Bastin

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Abstract

The paper 'A Homological Foundation for Scale Problems in Physics' (Atkin & Bastin, 1970) is criticised on account of several inconsistencies in the argument. Possible applications of the general ideas used are then discussed in the context of a 'quantum logic' type of framework.

1. *Introduction*

In a previous volume of this journal there appeared a paper (Atkin & Bastin, 1970) which introduced a discrete, combinatorial approach to physics. Unfortunately the ideas in this paper, many quite valuable, were almost entirely concealed by a number of obscurities in the text. In the present paper, therefore, I shall try to indicate the points where there appear to be gaps in the argument of Atkin and Bastin and to give a brief discussion of one possible application of their methods.

2. *The 'Base Complex'*

A fundamental role is played by a simplicial complex K and a related complex K^+ called the base complex. This is built up in time by a process called discrimination (Section 2),† whose operation in one dimension is described and which is supposed to continue to higher dimensions (p. 452, l. 9). In this part of the paper a 'complex' clearly means what I shall here call an *abstract complex*: a simplicial complex whose simplexes are in 1–1 correspondence with certain subsets of a basic set S^+, the dimension of the simplex being one less than the cardinality of the subset and the facing relation corresponding to inclusion of subsets.

The authors appear to mean that the discrimination process builds up the complex through a series of stages labelled with a dimension index p.

† References in parentheses to article numbers and to page and line numbers refer to Adkin & Bastin, 1970, hereafter cited as AB.

The pth stage consists of a number of steps; at each step there is added to the complex the set of simplexes corresponding to all *proper* subsets of a set of the form $\{\phi, a_1, \ldots, a_{p+1}\}$ where $\phi, a_1, \ldots, a_{p+1}$ are distinct members of S^+, and where ϕ is a distinguished point of S^+ called the antipoint.† The result after the pth stage is denoted by $K_p{}^+$ and the process is started by taking $K_0{}^+ = \{(a) | a \in S^+\}$ (where (a) is the 0-simplex corresponding to a). The observable p-cycles of $K_p{}^+$ are called *objects* and are described by the homology groups $H_p(K_p{}^+)$ with $p = 1, 2, \ldots, n = \text{card}(S^+) - 1$.

They continue:

> 'These developments lead us to identify a class of related physical objects with a set of *homology groups* $H_p(K^+)$ on a suitable complex (Hilton & Wylie, 1962) ... The complex K^+ ... possesses $\beta_0 + 1$ vertices and is embedded in an abstract polyhedron possessing these same vertices' (p. 453, l. 4).

Here $\beta_0 = n$. Note that K^+ is not defined and it is hard to see what definition it could have. Certainly it is not $\bigcup_{p=0}^{n} K_p{}^+ = K_n{}^+$ since in the progressive formation of this complex the formation of higher stages obliterates the homology of the lower stages, $H_p(K_n{}^+) \neq H_p(K_p{}^+)$. By only using the homology of the last complex the essential aspect of progressive growth would be lost, as is indeed emphasised in AB (p. 452, l. 9).

On the other hand, we could hardly suppose K^+ to be arbitrarily concocted so as to have the right homology. If that were the case we should be unable to credit it with any physical role at all. This leads to

Objection 1. Depending on the application, either K^+ should be identified with $K_p{}^+$ for some p, or the use of K^+ should be replaced by that of the entire sequence $\{K_p{}^+\}$. If this is not done K^+ cannot be linked to the discrimination process but should be introduced *de novo*.

In Section 4 of AB a particular K^+ is used to represent a 'common backcloth of observation', and is denoted by BK^+. It is referred to as the 'background complex' (p. 454, l. 11) or the 'base complex' (p. 459, l. 15). In this section *Objection* 1 becomes crucial because appeal is now made to the ring structure of $H^*(K^+)$, thus relating simplexes of different dimensions. To do this we must work within a *single* complex, not a sequence where a different complex is used for each dimension. Since this one complex is supposed to contain, through its homology groups, all the information about the objects in the various $K_p{}^+$ (cf. p. 454), its definition becomes of central importance.

In Section 5 a comparison with conventional measurement is sought. This involves interpreting BK^+ and then finding a link with 'continuous' measurement. The interpretation is difficult to follow, but this might be

† Here I use S^+ to denote the set $S \cup \phi$ (p. 451, l. 4). Later (p. 454, l. 9) AB use S for our S^+.

expected of the early stages in a radically new theory. Two ideas seem paramount:

(a) BK^+ is a theoretical entity underlying measurement processes, linked to observation only by the numbers which arise from taking Kronecker indices between cycles and cocycles (cf. p. 455);
(b) BK^+ could, at least sometimes, be an array of points in 1–1 correspondence with numbers obtained from some observation.

The latter idea is suggested by a reference to S as 'the set of observations' (p. 451, l. 20) and by the linking of 'the observed plane' to the base complex (p. 458, l. 5).

This double interpretation then becomes the source of a major mathematical confusion. We read (p. 459, l. 1):

> 'The plane is therefore $P_L{}^1 \times P_\lambda{}^1$, where (for example) P_L denotes the projective line specified by the grating length. Now, by the Kunneth Formula (Hilton & Wylie, 1962) we deduce the homology of base complex as
>
> $$H(P_L{}^1 \times P_\lambda{}^1) = H_0 \oplus H_1 \oplus H_2$$
>
> where $H_0 = J$, $H_1 = J \oplus J$, and $H_2 = J$.

Thus not only has a passage to an infinite number of points taken place, but the homology has changed from that of an abstract complex, which would give an increasing number of generators as more points were discriminated, to some sort of topological homology, Čech or singular simplicial. It is clear from the text immediately following this quotation that this new homology is to be understood in the same context as the old.

Objection 2. No explanation is given for the passage from the homology of an abstract complex to a Čech-like homology. Moreover the same notation in the same context is used for the two sorts.

This confusion is particularly illustrated by *Case* 1 on p. 457. We start with a complex BK^+ which, from what had preceded, would be assumed to be an abstract complex in the sense used here. A reference to '*the* basic 1-cycle' (my italics) and a comparison with article 2 suggests that BK^+ consists of three points ($S^+ = \phi, a, b$) and that a multiplicity of measurements are obtained from different cocycles of the form $n\hat{z}^1$ (p. 457, l. 10) for a generator $\hat{z}^1$ of the cocycle group (whose class presumably generates the cohomology group). The ring R of coefficients is required to be isomorphic to J (p. 456, l. 5) and is determined by an integer m as $R = m^{-1}J$, in the obvious way, since we have, for the dual $\hat{z}_1$ of $\hat{z}^1$, $(\hat{z}_1, \hat{z}^1) = m^{-1}$. Thus $m \neq 0$.

The trouble is that to obtain the projective line we must have a range of values for m. But each different m corresponds to a different experimental arrangement, with a different (though possibly isomorphic) base complex and a *different coefficient group*. These difficulties are slid over and, without any explanation, we find at the end of the argument in *Case* 1 that the following changes appear to have been made:

(i) m is allowed to be variable with the homology remaining fixed.
(ii) m is allowed to be zero (otherwise only the Euclidean line would obtain).
(iii) The base complex is now regarded as the (potentially) infinite set of observations obtained from the previous conception of the base complex, involving a shift in interpretation from (a) to (b).
(iv) The homology of the base complex is changed from that of an abstract complex to, probably, Čech homology with *integer* coefficients.

To all intents and purposes we are now in a different theory from that of article 2, and it is this new theory which is used from now on.

3. *The Nature of Memory*

One of the main virtues of the paper is that it tries to take the measuring and recording process fully into account, never going beyond the capabilities of the measuring and recording system being used; and that because of this certain coupling constants are held to arise naturally. This is described in Section 3. It is seen that the argument, far from giving any insight into the nature of the recording process, makes the apparently arbitrary assumption that there exists a measuring process whose memory store has size n^{2k} at the level k of the hierarchy, n being the number of vertices of K_0 at level 1. While this had some basis when the theory was presented in terms of vector spaces (Bastin, 1966) it has no foundation in the present theory.

4. *The Addition of Complexes*

On p. 463 there is reference to '... a base complex ... which may be denoted symbolically by $P_L{}^1 \times (P_\lambda{}^1 + M)$ ...'.

The symbol M, 'denoting the magnetic field' is presumably a manifold (like P^1) since there is a reference to $H^1(M)$, and $(P_\lambda{}^1 + M)$ denotes the result of combining M and $P_\lambda{}^1$ in such a way that the generators of $H^1(M)$ and $H^1(P_\lambda{}^1)$ becomes 'linearly dependent'. Without any definition of M or any indication as to how the combination is to be done or any proof that such a thing might be possible within the context of the theory, this section amounts to no more than the blind assertion that there is some way of fixing the base complex so as to get the right answers. This hardly constitutes either a test or an elucidation of the theory.

Similar remarks apply to the '+' sign in *Case* 5.

5. *Discussion*

From Section 2 of this note it would appear that in AB we have an amalgam of two distinct theories: Sections 2 and 3, based on Bastin, 1966, use one concept of K^+, while Sections 4 and 5, based on Atkin, 1971, use another. Until a more coherent unity can be obtained, the two theories must be judged separately on their own merits.

It is possible that the most fruitful application of these ideas will prove to be in quantum mechanics, and so I shall conclude with an indication of the general picture which may be involved here, departing somewhat, it must be admitted, from the basic philosophies of Atkin and Bastin.

Consider a 'quantum logic' type of situation, with a fixed system and a set M of measurements which may be performed upon it yielding either 0 or 1 as the result. Let us refer to an ensemble E of experimental events: each element of E labels some particular occasion on which the system is set up and measured. Suppose too that meaning can be attached to the measurement of an ordered set $\mathbf{m} = (m_a, m_2, \ldots, m_p) \in M^{p+1}$ on the same occasion $e \in E$, for each integral p, yielding a sequence in $\{0,1\}^{p+1}$ which we call $(e, \mathbf{m})$. We can now define an abstract complex $K(M)$ in the same way as the nerve of a covering is defined.

For each $p = 0, 1, 2, \ldots$ form the collection $\tilde{K}_p(M)$ of all subsets $\{m_a, \ldots, m_p\}$ of M such that for some ordering of the subset, giving $\mathbf{m}$, and for some $e \in E$, we have $(c, \mathbf{m}) = \mathbf{1} = (1, 1, \ldots, 1)$. Then to obtain $K(M)$ each element of $\tilde{K}(M) = \bigcup_{p=0}^{m} \tilde{K}_p(M)$ is given an arbitrary ordering to define simplexes whose facing relations are given in terms of inclusion relations between the corresponding members of $\tilde{K}(M)$ in the usual way.

This complex then has much in common with the base complex in AB. Specific states of the system give rise to cochains on K whose Kronecker indices with the chains yields the information corresponding to the nature of M; position, momentum or whatever it may be. If the system is a classical one, the members of M could correspond to the characteristic functions of a covering $\mathscr{C}$ of the phase space. Then $K(M)$ will be the nerve of $\mathscr{C}$ and the Čech homology of phase space is then connected with that of $K(M)$. In general a detailed discussion would be required of the way the homology groups behaved as the experimental situation was refined, to obtain a general analogue of Čech homology.

Note also that, just as direct products of complexes arose in AB, so here tensor products of chain groups arise. If we have two sets of measurements, M_1 and M_2 on a common E, then for $\mathbf{m}_1 \in M_1^{p+1}$ and $\mathbf{m}_2 \in M_2^{q+1}$ we may have results of the form $(e, \mathbf{m}_1, \mathbf{m}_2) \in \{0,1\}^{p+q+2}$, which lead in the same way to a bigraded double complex $K_{pq}(M_1, M_2)$ and hence to the corresponding single complex, based on pairs $(\mathbf{m}_1, \mathbf{m}_2)$ such that $(\exists e)((e, \mathbf{m}_1, \mathbf{m}_2) = 1)$, whose chain group is a subgroup of $C(K(M_1)) \otimes C(K(M_2))$. If M_1 and M_2 represent the performance of the same measurements at successive times, then these constructions carry information about the dynamics of the system.

We must recognise that the complexes used here are not arrived at by discrimination, and do not embody the interesting cumulative aspect of a discrimination complex, as described by Bastin (1966). It is hoped, however, that this brief example, together with the classical material of Atkin (1971) may suggest the possibilities which stem from the approach of these authors.

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